

Aspects of defects in integrable quantum field theory

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Abstract

Defects are ubiquitous in nature, for example dislocations, shocks, bores, or impurities of various kinds, and their descriptions are an important part of any physical theory. However, one might ask the question: what types of defect are allowed and what are their properties if it is required to maintain integrability within an integrable field theory in two-dimensional space-time? This talk addresses a collection of ideas and questions including examples of integrable defects and the curiously special roles played by energy-momentum conservation and Bäcklund transformations, solitons scattering with defects and some interesting effects within the sine-Gordon model, defects in integrable quantum field theory and the construction of transmission matrices, and concluding with remarks on algebraic considerations and future directions.

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1 An integrable discontinuity

Start with a single selected point on the x -axis, say $x_0 = 0$, and denote the field to the left of it ($x < x_0$) by u , and to the right ($x > x_0$) by v :

$$\cdots \text{---} \underset{u(x,t)}{\text{---}} \bullet \underset{v(x,t)}{\text{---}} \cdots$$

x_0

The field equations in the two separated domains are:

$$\partial^2 u = -\frac{\partial U}{\partial u}, \quad x < x_0, \quad \partial^2 v = -\frac{\partial V}{\partial v}, \quad x > x_0.$$

If U, V both represent potentials for integrable models, the question really concerns how the fields u, v may be ‘sewn’ together at x_0 to preserve integrability? There may be several kinds of sewing condition and each would represent a specific type of defect. An elaboration would also allow some degrees of freedom attached to the point $x = 0$ and allowed to interact with both fields. That possibility will be considered later. If there is no additional degree of freedom, the defect will be called ‘type I’. The fields u, v could be multicomponent but for most of the time in this talk they will refer to single component scalar fields. If there is just one defect it will be placed at $x_0 = 0$ from now on.

One natural choice (a δ -impurity) would be to have the two fields continuous across $x = 0$ but allow a discontinuity in the first spatial derivative. For example, put

$$u(0, t) = v(0, t), \quad u_x(0, t) - v_x(0, t) = \mu F(u(0, t)),$$

but, while there are likely to be interesting effects the integrability is actually lost [1]. To try to make some progress, note there is clearly a distinguished point, translation symmetry is lost and the conservation laws - at least some of them - (for example, energy-momentum), are violated unless the impurity contributes compensating terms. The problem can be restated in terms of seeking the suitable compensating associated with the defect to ensure there are enough conserved quantities to guarantee integrability.

Consider first the field contributions to energy-momentum:

$$P^\mu = \int_{-\infty}^0 dx T^{0\mu}(u) + \int_0^\infty dx T^{0\mu}(v).$$

Using the field equations, can we arrange

$$\frac{dP^\mu}{dt} = -[T^{1\mu}(u)]_{x=0} + [T^{1\mu}(v)]_{x=0} = -\frac{dD^\mu(u, v)}{dt},$$

with the right hand side depending only on the fields at $x = 0$? If so, $P^\mu + D^\mu$ will be conserved with D^μ being the defect contribution. Only a few possible sewing conditions (and bulk potentials U, V) are permitted for this to work.

Note, it might be more natural to consider higher spin charges and indeed that was the historical route [2]. However, for reasons that are not yet entirely clear it appears to be enough to investigate the conservation of momentum alone, since that already produces constraints equivalent to those obtained by other means.

Consider first the field contributions to energy and calculate

$$\frac{dP^0}{dt} = [u_x u_t]_0 - [v_x v_t]_0.$$

Setting $u_x = v_t + X(u, v)$, $v_x = u_t + Y(u, v)$, clearly

$$\frac{dP^0}{dt} = u_t X - v_t Y.$$

This is a total time derivative if

$$X = -\frac{\partial D^0}{\partial u}, \quad Y = \frac{\partial D^0}{\partial v},$$

for some D^0 . Then

$$\frac{dP^0}{dt} = -\frac{dD^0}{dt}.$$

This was only to be expected, with no further constraints on $U(u), V(v)$ or D^0 , since time translation has not been deliberately violated.

On the other hand, for momentum

$$\begin{aligned} \frac{dP^1}{dt} &= - \left[\frac{u_t^2 + u_x^2}{2} - U(u) \right]_{x=0} + \left[\frac{v_t^2 + v_x^2}{2} - V(v) \right]_{x=0} \\ &= \left[-v_t X + u_t Y - \frac{X^2 - Y^2}{2} + U - V \right]_{x=0} = -\frac{dD^1}{dt}. \end{aligned}$$

This is a total time derivative of a functional of the fields at $x = 0$ provided the first piece is a perfect differential and the second piece vanishes. Thus

$$X = -\frac{\partial D^0}{\partial u} = \frac{\partial D^1}{\partial v}, \quad Y = \frac{\partial D^0}{\partial v} = -\frac{\partial D^1}{\partial u},$$

and

$$\frac{\partial^2 D^0}{\partial v^2} = \frac{\partial^2 D^0}{\partial u^2}, \quad \frac{1}{2} \left(\frac{\partial D^0}{\partial u} \right)^2 - \frac{1}{2} \left(\frac{\partial D^0}{\partial v} \right)^2 = (U - V),$$

which is strongly constraining. What are the possible combinations U, V, D^0 ? It is not difficult to check that sine-Gordon, Liouville, massless free and massive free are in fact the only possibilities. For example, if $U(u) = m^2 u^2 / 2$, $V(v) = m^2 v^2 / 2$, D^0 turns out to be

$$D^0(u, v) = \frac{m\sigma}{4}(u+v)^2 + \frac{m}{4\sigma}(u-v)^2,$$

and σ is a free parameter.

Note though, the other single field integrable theory of a similar type, the Tzitzéica potential (otherwise known as Bullough-Dodd, Mikhailov-Zhiber-Shabat, or $a_2^{(2)}$ affine Toda),

$$U(u) = e^u + 2e^{-u/2},$$

is not a possible solution. This was mysterious for a while and the resolution will come later when a more general type of defect will be explored.

Note also, there is a Lagrangian description of this type of defect (type I) [2]:

$$\mathcal{L} = \theta(-x)\mathcal{L}(u) + \delta(x) \left(\frac{uv_t - u_tv}{2} - D^0(u, v) \right) + \theta(x)\mathcal{L}(v).$$

2 sine-Gordon

Choosing u, v to be sine-Gordon fields (and scaling the coupling and mass parameters to unity), then:

$$D^0(u, v) = 2 \left(\sigma \cos \frac{u+v}{2} + \sigma^{-1} \cos \frac{u-v}{2} \right)$$

to find

$$\begin{aligned} x < x_0 : \quad & \partial^2 u = -\sin u; & x > x_0 : \quad & \partial^2 v = -\sin v, \\ x = x_0 : \quad & u_x = v_t - \sigma \sin \frac{u+v}{2} - \sigma^{-1} \sin \frac{u-v}{2}, \\ x = x_0 : \quad & v_x = u_t + \sigma \sin \frac{u+v}{2} - \sigma^{-1} \sin \frac{u-v}{2}. \end{aligned}$$

Unexpectedly, the last two expressions are recognised to be a Bäcklund transformation ‘frozen’ at $x = 0$, already a signal of integrability. In fact, in this case a Lax pair can be devised to incorporate the defect and generate suitably modified higher spin conserved charges. So the system is integrable in the standard way [2].

Consider a soliton incident from $x < 0$. It will not be possible to satisfy the sewing conditions (in general) unless a similar soliton emerges into the region $x > 0$. Thus, using convenient expressions for the standard soliton solutions

$$e^{iu/2} = \frac{1+iE}{1-iE}, \quad e^{iv/2} = \frac{1+izE}{1-izE}, \quad E = e^{ax+bt+c}, \quad a = \cosh \theta, \quad b = -\sinh \theta,$$

where z is to be determined. It is also useful to set $\lambda = e^{-\eta}$ and then find the sewing conditions imply

$$z = \coth \left(\frac{\eta - \theta}{2} \right).$$

This result has several interesting consequences [2, 3]:

- $\theta > \eta$ implies $z < 0$, meaning the soliton emerges as an anti-soliton and the final state will contain a discontinuity of magnitude 4π at $x = 0$;
- $\theta = \eta$ implies $z = \infty$ and there is no emerging soliton, meaning the energy-momentum of the soliton is captured by the ‘defect’, as is its topological charge indicated by a discontinuity 2π in the final state;
- $\theta < \eta$ implies $z > 0$ meaning the soliton retains its character, and the final state does not acquire an additional discontinuity.

It's worth making a few other comments:

- Defects at $x = x_1 < x_2 < x_3 < \dots < x_n$ behave independently, each contributes a factor z_i for a total 'delay' $z = z_1 z_2 \dots z_n$;
- Each component of a multisoliton is affected separately, thus at most one of them can be 'filtered out' (because the rapidities of individual soliton components must be different);
- Since a soliton can be absorbed, can a starting configuration with $u = 0$, $v = 2\pi$ decay into a soliton? This would seem to need quantum mechanics to provide a probability for decay after a specific time.
- The properties listed above suggest that a type I defect could be used to manipulate solitons, see for example [4].
- Extending the type I defect to other field theories of affine Toda type is only possible for the series of models based on $a_n^{(1)}$ [5, 6] .

3 Classical type II defect

Since not all field theories of affine Toda type can accommodate a type I defect it is natural to try adding extra degrees of freedom to the defect itself [7]. For example, consider two relativistic field theories with fields u and v , and add $\lambda(t)$ at the defect location as follows:

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x) (2q\lambda_t - D^0(\lambda, p, q)) ,$$

where

$$q = \left. \frac{u-v}{2} \right|_0, \quad p = \left. \frac{u+v}{2} \right|_0.$$

Then the usual steps lead to equations of motion:

$$\partial^2 u = -\frac{\partial U}{\partial u}, \quad x < 0, \quad \partial^2 v = -\frac{\partial V}{\partial v}, \quad x > 0,$$

alongside defect conditions at $x = 0$:

$$2q_x = -D_p^0, \quad 2p_x - 2\lambda_t = -D_q^0, \quad 2q_t = -D_\lambda^0.$$

Note, if there was no defect potential λ would act as a lagrange multiplier.

As before, consider momentum

$$P^1 = - \int_{-\infty}^0 dx u_t u_x - \int_0^\infty dx v_t v_x,$$

and seek a functional $D^1(u, v, \lambda)$ such that $P_t^1 \equiv -D_t^1$. Then $P^1 + D^1|_{x=0}$ will be the total conserved momentum of the system. This leads to the following constraints on U, V, D^1 :

$$D_p^0 = D_\lambda^1, \quad D_\lambda^0 = D_p^1, \quad D_\lambda^1 D_q^0 - D_q^1 D_\lambda^0 = 2(U - V),$$

implying

$$D^0 = f(p + \lambda, q) + g(p - \lambda, q), \quad D^1 = f(p + \lambda, q) - g(p - \lambda, q),$$

with

$$f_\lambda g_q - g_\lambda f_q = U(u) - V(v).$$

The latter is also a powerful constraint since the left hand side depends on λ while the right hand side does not. However, this time it is not totally clear what the full set of solutions is. Nevertheless, it is certainly possible to choose f, g in such a way that the potentials U, V can be any one of sine-Gordon, Liouville, Tzitzéica, or free massive or massless; ie any one of the full set of single field, relativistic, integrable field theories. For example, for Tzitzéica:

$$U(u) = (e^u + 2e^{-u/2} - 3), \quad V(v) = (e^v + 2e^{-v/2} - 3)$$

the corresponding defect potential is given by

$$D^0 = 2\sigma \left(e^{(p+\lambda)/2} + e^{-(p+\lambda)/4} (e^{q/2} + e^{-q/2}) \right) + \frac{1}{\sigma} \left(8e^{-(p-\lambda)/4} + e^{(p-\lambda)/2} (e^{q/2} + e^{-q/2})^2 \right).$$

But note, the defect conditions following from this are not a ‘frozen’ Bäcklund transformation. Moreover, other multi-component affine Toda field theories can support type II defects, though not all of them (the missing ones are those based on root data corresponding to Kac-Dynkin diagrams containing a simple root with three connected neighbours - such as the e -series), which suggests there might be a type III Lagrangian waiting to be found. In the sine-Gordon model the type-II defect is also new - in a sense it is really two ‘fused’ type-I defects - and the corresponding defect conditions are not quite a frozen Bäcklund transformation even in that case.

4 Defects in quantum field theory

Based on facts gleaned from the classical soliton-defect scattering the following might be expected:

- soliton-defect scattering - pure transmission compatible with the bulk S-matrix and topological charge will be preserved but may be exchanged with the defect;
- for each type of defect there may be several types of transmission matrix (eg in sine-Gordon expect two different transmission matrices since the topological charge on a defect can only change by ± 2 when a soliton/anti-soliton passes).
- not all transmission matrices need be unitary - in the sine-Gordon model one turns out to be a resonance of the other.

Then, seek to understand different types of defect, assemblies of defects, defect-defect scattering, fusing defects, and so on.

The most relevant quantity will be the ‘transmission matrix’, denoted,

$$T_{a\alpha}^{b\beta}(\theta, \eta), \quad a + \alpha = b + \beta,$$

where α, β and a, b are defect and soliton labels, respectively, and η is a collection of defect parameters. Besides any general considerations, such as unitarity (where appropriate), or crossing properties, the transmission matrix will have to be compatible with the bulk S-matrix, suggesting the following algebraic constraints [8]:

$$S_{ab}^{cd}(\Theta) T_{d\alpha}^{f\beta}(\theta_a) T_{c\beta}^{e\gamma}(\theta_b) = T_{b\alpha}^{d\beta}(\theta_b) T_{a\beta}^{c\gamma}(\theta_a) S_{cd}^{ef}(\Theta).$$

Here $\Theta = \theta_a - \theta_b$ and sums are to be understood over the ‘internal’, repeated labels β, c, d .

Generally, for affine Toda field theories based on the root data of a Lie algebra g , a number of different transmission matrices are expected (all of them infinite-dimensional), labelled by elements of the associated weight lattice differing by roots. Thus the two possibilities for sine-Gordon, labelled by even or odd integers, respectively, might be more appropriately considered as labeled by integer or half-odd-integer spins.

Solution for type I

For sine-Gordon a solution was found some time ago by Konik and LeClair [9]. To describe their result, recall first how the Zamolodchikov S-matrix [10] depends on the rapidity variables θ and the bulk coupling β via

$$x = e^{\gamma\theta}, \quad q = e^{i\pi\gamma}, \quad \gamma = \frac{8\pi}{\beta^2} - 1;$$

it is also useful to define the variable

$$Q = e^{4\pi^2 i/\beta^2} = \sqrt{-q}.$$

Then, the K-L solution, written as a two by two matrix carrying the soliton labels, has the form

$$T_{a\alpha}^{b\beta}(\theta) = f(q, x) \begin{pmatrix} Q^\alpha \delta_\alpha^\beta & q^{-1/2} e^{\gamma(\theta-\eta)} \delta_\alpha^{\beta-2} \\ q^{-1/2} e^{\gamma(\theta-\eta)} \delta_\alpha^{\beta+2} & Q^{-\alpha} \delta_\alpha^\beta \end{pmatrix}$$

where $f(q, x)$ is not uniquely determined but, for a unitary transmission matrix should satisfy

$$\bar{f}(q, x) = f(q, qx), \quad f(q, x)f(q, qx) = (1 + e^{2\gamma(\theta-\eta)})^{-1}.$$

The K-L ‘minimal’ solution to these requirements has the following form

$$f(q, x) = \frac{e^{i\pi(1+\gamma)/4}}{1 + ie^{\gamma(\theta-\eta)}} \frac{r(x)}{\bar{r}(x)},$$

where it is convenient to put $z = i\gamma(\theta - \eta)/2\pi$ and

$$r(x) = \prod_{k=0}^{\infty} \frac{\Gamma(k\gamma + 1/4 - z)\Gamma((k+1)\gamma + 3/4 - z)}{\Gamma((k+1/2)\gamma + 1/4 - z)\Gamma((k+1/2)\gamma + 3/4 - z)}.$$

Remarks (taking $\theta > 0$): it is tempting to suppose η (possibly renormalized) is essentially the same parameter as appeared in the type I model of a defect, given the apparently similar features to the classical soliton-defect scattering [3]. In particular:

- $\eta < 0$ - the off-diagonal entries dominate;
- $\theta > \eta > 0$ - the off-diagonal entries dominate;
- $\theta < \eta > 0$ - the diagonal entries dominate.

The particular rapidity $\theta = \eta$, which had a special meaning in the classical picture, is not so special here. Nevertheless, the function $f(q, x)$ has a simple pole nearby at

$$\theta = \eta - \frac{i\pi}{2\gamma} \rightarrow \eta, \quad \beta \rightarrow 0,$$

and this pole is like a resonance, with complex energy,

$$E = m_s \cosh \theta = m_s (\cosh \eta \cos(\pi/2\gamma) - i \sinh \eta \sin(\pi/2\gamma)).$$

Clearly, it has a ‘width’ proportional to $\sin(\pi/2\gamma)$, which tends to zero as $\beta \rightarrow 0$. The fact solitons are treated differently to anti-solitons in the diagonal entries of the K-L solution also has its origins in the Lagrangian piece linear in time derivatives, a feature seen in semi-classical arguments.

The Zamolodchikov S-matrix has ‘breather’ poles corresponding to soliton-anti-soliton bound states at

$$\Theta = i\pi(1 - n/\gamma), \quad n = 1, 2, \dots, n_{\max} < \gamma;$$

and the bootstrap can be used to calculate the transmission factors for breathers. For the lightest breather the result is:

$$T(\theta) = -i \frac{\sinh\left(\frac{\theta-\eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta-\eta}{2} + \frac{i\pi}{4}\right)},$$

which has precisely the form of the transmission factor for a plane wave encountering a defect in the linearised sine-Gordon model.

Solution for type II

Apart from the solution obtained by Konik and LeClair, there is another, more general, set of solutions to the quadratic compatibility relations for the transmission matrix [11]:

$$T_{a\alpha}^{b\beta}(\theta) = \rho(\theta) \begin{pmatrix} (a_+ Q^\alpha + a_- Q^{-\alpha} x^2) \delta_\alpha^\beta & x (b_+ Q^\alpha + b_- Q^{-\alpha}) \delta_\alpha^{\beta-2} \\ x (c_+ Q^\alpha + c_- Q^{-\alpha}) \delta_\alpha^{\beta+2} & (d_+ Q^\alpha x^2 + d_- Q^{-\alpha}) \delta_\alpha^\beta \end{pmatrix}$$

where $x = e^{\gamma\theta}$ and the free constants satisfy the two constraints

$$a_\pm d_\pm - b_\pm c_\pm = 0.$$

These and $\rho(\theta)$ are constrained further by crossing and unitarity. More details are provided in ref[11] but a few remarks are in order:

- for a choice of parameters similarity with classical scattering suggests this describes a type II defect;
- with $a_- = d_+ = 0$ and $b_+ = c_- = 0$ or $b_- = c_+ = 0$ (and after a similarity transformation), it reduces to the type I solution;
- for certain other choices of parameters it reduces to a direct sum of the Zamolodchikov S-matrix itself and two infinite dimensional pieces, suggesting that a type II defect is itself similar to a soliton.

Solution for the Tzitzéica model

Recall: the Tzitzéica quantum field theory is a little different to its classical version in so far as it possesses mass-degenerate solitons with charges ± 1 and 0 whose scattering is described by the Izergin-Korepin-Smirnov S-matrix [12]. Using the latter and solving the compatibility relations leads to [13]

$$T_{a\alpha}^{b\beta}(\theta) = \rho(\theta) \begin{pmatrix} (\varepsilon^2 q^{2\alpha} + \tau^2 q^{-2\alpha} x) \delta_\alpha^\beta & \varepsilon \mu(\alpha) \delta_\alpha^{\beta-1} & M(\alpha) \delta_\alpha^{\beta-2} \\ \tau \lambda(\alpha) x \delta_\alpha^{\beta+1} & (\tilde{\tau} \varepsilon + \tau \tilde{\varepsilon} x) \delta_\alpha^\beta & \tilde{\tau} \mu(\alpha) q^{-2\alpha-1} \delta_\alpha^{\beta-1} \\ L(\alpha) x \delta_\alpha^{\beta+2} & \tilde{\varepsilon} q^{2\alpha-1} \lambda(\alpha) x \delta_\alpha^{\beta+1} & (\tilde{\varepsilon}^2 q^{2\alpha} x + \tilde{\tau}^2 q^{-2\alpha}) \delta_\alpha^\beta \end{pmatrix}$$

where

$$M(\alpha) = \mu(\alpha) \mu(\alpha+1) \frac{q^{-2\alpha-1}}{1+q^2}, \quad L(\alpha) = \lambda(\alpha) \lambda(\alpha-1) \frac{q^{2\alpha-1}}{1+q^2},$$

$$\mu(\alpha) \lambda(\alpha+1) = (q + q^{-1}) (\tau \tilde{\tau} q^{-2\alpha-1} + \varepsilon \tilde{\varepsilon} q^{2\alpha+1}), \quad q = e^{i4\pi^2/\beta^2}.$$

As before, the overall factor and some combinations of constants are constrained by insisting on crossing and a consistent bootstrap (in this case the model does not possess unitary scattering but each soliton is a bound state of two others). Thus,

$$\rho(\theta) \rho(\theta i\pi) (\tilde{\tau} \varepsilon + \tau \tilde{\varepsilon} q^{-4} x) (\tilde{\tau} \varepsilon - \tau \tilde{\varepsilon} q^{-2} x) = 1, \quad \rho(\theta) = (\tilde{\tau} \varepsilon + \tau \tilde{\varepsilon} x) \rho(\theta + i\pi/3) \rho(\theta - i\pi/3).$$

This solution looks complicated but it can be approached in an alternative manner suggested by Weston [14] and used in the sine-Gordon case to reproduce the type II transmission matrix described above. The idea is to associate the transmission matrix with an intertwiner linking the two co-products of a finite dimensional and an infinite-dimensional representation of a Borel subalgebra of the ‘quantised’ affine algebra $U_q(a_2^{(2)})$. The solitons belong to the finite dimensional representation while the infinite-dimensional representation describes the defect. Moreover it is convenient to describe the infinite dimensional representation in terms of operators that create and annihilate topological charge. First, a short introduction to the algebra might be useful.

With α_0 the shorter root, the Cartan matrix of $a_2^{(2)}$ is

$$C_{ij} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \quad i, j = 0, 1.$$

The quantised algebra $U_q(a_2^{(2)})$ introduced by Drinfel’d and Jimbo (see [15]) has six generators

$$\{X_1^\pm, X_0^\pm, K_1, K_0\},$$

satisfying:

$$\begin{aligned} [K_1, K_0] &= 0, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad i = 0, 1, \\ K_0 X_0^\pm K_0^{-1} &= q^{\pm 4} X_0^\pm, \quad [X_0^+, X_0^-] = \frac{K_0^2 - K_0^{-2}}{q^4 - q^{-4}}, \\ K_1 X_1^\pm K_1^{-1} &= q^{\pm 1} X_1^\pm, \quad [X_1^+, X_1^-] = \frac{K_1^2 - K_1^{-2}}{q - q^{-1}}, \\ K_1 X_0^\pm K_1^{-1} &= q^{\mp 2} X_0^\pm, \quad K_0 X_1^\pm K_0^{-1} = q^{\mp 2} X_1^\pm, \quad [X_1^\pm, X_0^\mp] = 0, \end{aligned}$$

and the Serre relations,

$$\begin{aligned} \sum_{k=0}^5 (-1)^k \begin{bmatrix} 5 \\ k \end{bmatrix}_q (X_1^\pm)^{5-k} X_0^\pm (X_1^\pm)^k &= 0, \\ \sum_{k=0}^2 (-1)^k \begin{bmatrix} 2 \\ k \end{bmatrix}_{q^4} (X_0^\pm)^{2-k} X_1^\pm (X_0^\pm)^k &= 0. \end{aligned}$$

The coproducts Δ, Δ' are given by:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(X_i^\pm) = X_i^\pm \otimes K_i^{-1} + K_i \otimes X_i^\pm, \quad i = 0, 1,$$

and

$$\Delta'(K_i) = \Delta(K_i), \quad \Delta'(X_i^\pm) = K_i^{-1} \otimes X_i^\pm + X_i^\pm \otimes K_i, \quad i = 0, 1.$$

The fundamental representation of this algebra is three-dimensional and the intertwiner between the two co-products of three-dimensional representations is the S-matrix.

For our purposes, an infinite-dimensional representation of the Borel subalgebra generated by $\{X_1^+, X_0^+, K_1, K_0\}$ is also required and conveniently realised using a pair of annihilation and creation operators similar to those introduced by Macfarlane and Biedenharn twenty years ago [16]. In detail, they are defined by

$$a|j\rangle = F(j)|j-1\rangle, \quad \hat{a}|j\rangle = |j+1\rangle, \quad N|j\rangle = j|j\rangle, \quad j \in \mathbb{Z}.$$

Note, $F(j)$ need not vanish for any j , since j represents a topological charge. Also,

$$a\hat{a} = F(N+1), \quad \hat{a}a = F(N), \quad aG(N) = G(N+1)a, \quad \hat{a}G(N) = G(N-1)\hat{a}$$

where $G(N)$ is any function of the number operator. It is not necessary to insist on the conjugation relation $\hat{a} = a^\dagger$. Using these, the generators of the Borel subalgebra are taken to be:

$$X_1^+ = \hat{a}, \quad X_0^+ = a a, \quad K_1 = \kappa_1 q^N, \quad K_0 = \kappa_0 q^{-2N},$$

where κ_0 and κ_1 are constants. This choice satisfies the Borel sub-algebra and the Serre relations require,

$$\begin{aligned} \sum_{k=0}^5 (-1)^k \begin{bmatrix} 5 \\ k \end{bmatrix}_q F(N+k)F(N+k+1) &= 0, \\ \sum_{k=0}^2 (-1)^k \begin{bmatrix} 2 \\ k \end{bmatrix}_{q^4} F(N+2k) &= 0, \end{aligned}$$

which in turn require

$$\begin{aligned} \hat{a}a = F(N) &= (b_1 (-)^N + c_1) q^{-2N} + (b_2 (-)^N + c_2) q^{2N} \\ b_1 c_2 &= b_2 c_1. \end{aligned}$$

Finally, it is necessary to use a homogeneous gradation [17], namely

$$E_1 = X_1^+, \quad F_1 = X_1^-, \quad E_0 = x X_0^+, \quad F_0 = x^{-1} X_0^-.$$

Then the transmission matrix T is an intertwiner of the infinite dimensional representation with space \mathcal{V} and the three-dimensional representation with space V :

$$T(z/x) : \mathcal{V}_z \otimes V_x \rightarrow \mathcal{V}_z \otimes V_x$$

achieved by solving the linear condition (for any element b of the Borel subalgebra),

$$T \Delta(b) = \Delta'(b) T,$$

to find

$$T = \begin{pmatrix} a' q^{-2N} + a'' q^{2N} & k q^N a & v a a \\ j q^{-N} \hat{a} & b & i q^{-N} a \\ w \hat{a} \hat{a} & l q^N \hat{a} & c' q^{2N} + c'' q^{-2N} \end{pmatrix}$$

where the coefficients have to be chosen suitably [13].

5 Discussion

There is no reason why defects of type I or type II should be static. In fact, for type I, travelling defects have been analysed within the sine-Gordon theory [3], and, if there are several moving at different speeds their classical scattering is consistent with their interactions with solitons owing to the ‘permutability theorem’ for repeated Bäcklund transformations. On the other hand, their quantum scattering is not yet completely determined, though there is a candidate S-matrix compatible with the soliton transmission matrix [3, 14, 18]. There are many interesting features pertaining to sine-Gordon defects that have been examined by other people, for example refs[19, 20], and also to defects that might occur within extensions to sine-Gordon including fermions or analysing the relationship with the massive Thirring model [21], or within other integrable field theories not of Toda type, such as NLS, KdV or mKdV [22]. Defects can be constructed within the $a_r^{(1)}$ series of affine Toda models [5] and transmission matrices have been written down for both types I and II [6, 13]. Generalised annihilation and creation operator representations of Borel subalgebras for all affine quantum groups can be constructed [13], though there are alternatives arising in other applications: for example, to construct representations of Baxter’s Q -operator in the context of solvable spin chains [23].

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